The Non-quasi-static monochromatic eigenstates of Maxwell's equations in a two-constituent three flat slabs microstructure and their application for calculating the local electric field of an oscillating electric dipole

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Background

- Materials with negative refractive index have been shown to enable the generation of an enhanced resolution image where both propagating and non-propagating waves are employed [1],[2].
- Recent analysis of the imaging of a slab in a medium in the quasistatic regime showed that the maximum concentration of electric field occurs not at the geometric optics foci but at the interfaces between the negative permittivity slab and the positive permittivity slabs [3],[4].

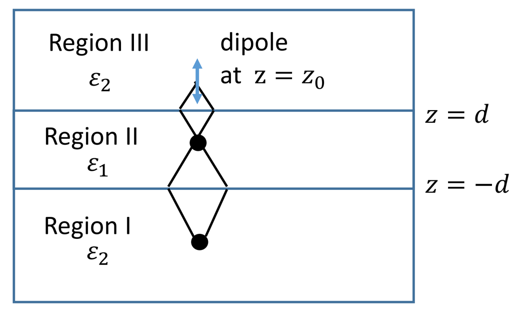
Goals

- Construct Eigenstates of monochromatic non-quasistatic Maxwell's equations for a setup of a slab in a medium.
- Calculate the eigenvalues.
- Express the physical Electric field in terms of these eigenstates.

Method

We consider a two-constituent microstructure of a ϵ_1 slab in a ϵ_2 medium

We assume $\mu = 1$ everywhere and solve the full Maxwell's equations where ϵ_1 and ϵ_2 can take any value. Electric dipole is situated at $z = z_0$.



Assuming that all the EM fields are monochromatic Maxwell's equations become, in Gaussian units,

$$\nabla\cdot(\epsilon\mathbf{E}) = 0,\ \nabla\times\mathbf{E} = \frac{i\omega}{c}\mathbf{H},\ \nabla\cdot\mathbf{H} = 0,\ \nabla\times\mathbf{H} = -\frac{i\omega}{c}\epsilon\mathbf{E} + \frac{4\pi}{c}\mathbf{J}.$$

We write

$$\epsilon(\mathbf{r}) = \epsilon_1\theta_1(\mathbf{r}) + \epsilon_2\theta_2(\mathbf{r}),$$

where $\theta_i(\mathbf{r})$, $i = 1, 2$, is a step function equal to 1 when $\mathbf{r}$ is inside the ϵ_i constituent and equal to 0 elsewhere.

From these we can obtain the following equation for the local electric field $\mathbf{E}(\mathbf{r})$:

$$-\nabla\times(\nabla\times\mathbf{E}) + k_2^2\mathbf{E} = uk_2^2\theta_1\mathbf{E} - \frac{4\pi i\omega}{c^2}\mathbf{J}_{\mathrm{dipole}}, \qquad (1)$$

$$u \equiv 1 - \frac{\epsilon_1}{\epsilon_2},\quad k_2^2 \equiv \epsilon_2\frac{\omega^2}{c^2},\quad \mathbf{J}_{\mathrm{dipole}} = -i\omega\hat{z}p\delta^3(\mathbf{r} - \mathbf{z}_0), \qquad (2)$$

where p is the *electric dipole moment*.

The last differential equation can be transformed into an integral equation by using a tensor Green function $G_{\alpha\beta}(\mathbf{r}, \mathbf{r}', k_2)$, defined by

$$-\nabla\times(\nabla\times\overleftrightarrow{G}) + k_2^2\overleftrightarrow{G} = k_2^2\hat{I}\delta^3(\mathbf{r} - \mathbf{r}'), \qquad \hat{I}_{\alpha\beta} \equiv \delta_{\alpha\beta}$$

with boundary conditions of an outgoing wave at large values of $|z|$.

The tensor Green's function takes the form

$$G_{\alpha\beta}(\mathbf{R}, k_2) = -\left(k^2\delta_{\alpha\beta} + \nabla_\alpha\nabla_\beta\right)G(\mathbf{R}, k_2),$$

where

$$G(\mathbf{R}, k) = \frac{1}{4\pi}\int_0^\infty q_\perp dq_\perp J_0(q_\perp\rho)\frac{e^{-\sqrt{q_\perp^2 - k^2}|R_z|}}{\sqrt{q_\perp^2 - k^2}}.$$

Using $\overleftrightarrow{G}(\mathbf{r} - \mathbf{r}', k_2)$ we can now "solve" Eq. (1) by treating its rhs as if it were known. In this way we get the following integral equation for the local electric field $\mathbf{E}(\mathbf{r})$:

$$\mathbf{E} = \mathbf{E_0} + u\hat{\Gamma}\mathbf{E},$$

$$\hat{\Gamma}\mathbf{E} \equiv \int dV'\theta_1(\mathbf{r}')\overleftrightarrow{G}(\mathbf{r} - \mathbf{r}', k_2)\cdot\mathbf{E}(\mathbf{r}'),$$

where $\mathbf{E_0}$ is the dipole field in a uniform medium.

For the case of no dipole we get an eigenvalue equation

$$s_n\mathbf{E}_n = \hat{\Gamma}\mathbf{E}_n, \quad s_n \equiv \frac{1}{u_n} \equiv \frac{\epsilon_2}{\epsilon_2 - \epsilon_{1,n}}.$$

The scalar product of two vector fields $\mathbf{F}(\mathbf{r})$, $\mathbf{E}(\mathbf{r})$ is now defined by

$$\langle\mathbf{F}|\mathbf{E}\rangle \equiv \int dV\theta_1(\mathbf{r})\mathbf{F}^*(\mathbf{r})\cdot\mathbf{E}(\mathbf{r}).$$

Under this definition $\hat{\Gamma}$ is a symmetric operator, because $G_{\alpha\beta}(\mathbf{R}, k) = G_{\beta\alpha}(-\mathbf{R}, k)$, but it is non-Hermitian because $\overleftrightarrow{G}(\mathbf{r} - \mathbf{r}', k_2)$ is complex. Therefore the left eigenstate of Γ, $\langle\tilde{\mathbf{E}}_{\mathbf{k}}^{\pm}|$ differs from its right eigenstate $|\mathbf{E}_{\mathbf{k}}^{\pm}\rangle$ though they have the same eigenvalue.

The unit operator can be expanded in terms of those states and their duals

$$\hat{1} = \sum_n\frac{|\tilde{\mathbf{E}}_n\rangle\langle\mathbf{E}_n|}{\langle\tilde{\mathbf{E}}_n|\mathbf{E}_n\rangle}.$$

We can now write the following formal solution of Eq. (1):

$$|\mathbf{E}\rangle = \frac{1}{1 - u\hat{\Gamma}}|\mathbf{E}_0\rangle = |\mathbf{E}_0\rangle + \frac{\hat{\Gamma}}{s - \hat{\Gamma}}|\mathbf{E}_0\rangle, \quad s \equiv \frac{1}{u},$$

and insert the unit operator to obtain

$$|\mathbf{E}\rangle - |\mathbf{E}_0\rangle = \sum_n\frac{s_n}{s - s_n}|\mathbf{E}_n\rangle\frac{\langle\tilde{\mathbf{E}}_n|\mathbf{E}_0\rangle}{\langle\tilde{\mathbf{E}}_n|\mathbf{E}_n\rangle}.$$

Constructing Eigenstates of Γ and the symmetry generators

- Symmetry under translation in x and y axes.
- Symmetry under reflection $z \to -z$.

$i\hbar\nabla_{xy}, \Pi_z$ commute with Γ ➡

We can construct eigenstates of $\Gamma, i\hbar\nabla_{xy}$ and Π_z.
The eigenstates satisfy

Even: $\mathbf{E}_{x,y}(-z) = \mathbf{E}_{x,y}(z)$, $\mathbf{E}_z(-z) = -\mathbf{E}_z(z)$
or
Odd: $\mathbf{E}_{x,y}(-z) = -\mathbf{E}_{x,y}(z)$, $\mathbf{E}_z(-z) = \mathbf{E}_z(z)$

Results

TM modes and their eigenvalues

$$\mathbf{E}_{\mathbf{k}}^{+} = e^{i\mathbf{k}\cdot\rho}\begin{cases}\mathbf{A}^{+}e^{-ik_{2z}z} & \mathbf{r}\in\mathrm{I}\\ B_z^{+}\sin(k_{1z}z)\,\mathbf{e}_z + B_{\mathbf{k}}^{+}\cos(k_{1z}z)\,\mathbf{e}_{\mathbf{k}} & \mathbf{r}\in\mathrm{II}\\ -A_z^{+}e^{ik_{2z}z}\mathbf{e}_z + A_{\mathbf{k}}^{+}e^{ik_{2z}z}\mathbf{e}_{\mathbf{k}} & \mathbf{r}\in\mathrm{III}\end{cases}$$

$$\mathbf{E}_{\mathbf{k}}^{-} = e^{i\mathbf{k}\cdot\rho}\begin{cases}\mathbf{A}^{-}e^{-ik_{2z}z} & \mathbf{r}\in\mathrm{I}\\ B_z^{-}\cos(k_{1z}z)\,\mathbf{e}_z + B_{\mathbf{k}}^{-}\sin(k_{1z}z)\,\mathbf{e}_{\mathbf{k}} & \mathbf{r}\in\mathrm{II}\\ A_z^{-}e^{ik_{2z}z}\mathbf{e}_z - A_{\mathbf{k}}^{-}e^{ik_{2z}z}\mathbf{e}_{\mathbf{k}} & \mathbf{r}\in\mathrm{III}\end{cases}$$

We impose $\nabla\cdot\mathbf{E}_{\mathbf{k}} = 0$ and continuity of $E_{\mathbf{k}}, D_z$ at the interfaces, and thus get a system of linear homogeneous algebraic equations for the **A** and **B** coefficients. These have a nonzero solution only if ϵ_1 and ϵ_2 satisfy one of the following two equations

$$\epsilon_2 = \frac{i\epsilon_1^{+}k_{2z}\tan\left(d\sqrt{k_0^2\epsilon_1^{+} - |\mathbf{k}|^2}\right)}{\sqrt{k_0^2\epsilon_1^{+} - |\mathbf{k}|^2}}, \qquad \epsilon_2 = -\frac{i\epsilon_1^{-}k_{2z}\cot\left(d\sqrt{k_0^2\epsilon_1^{-} - |\mathbf{k}|^2}\right)}{\sqrt{k_0^2\epsilon_1^{-} - |\mathbf{k}|^2}},$$

where k_0 is the free-space wavenumber and $2d$ is the slab thickness.

TM modes inner products

Analytic expressions are obtained for the inner products.

$$\langle\mathbf{E}_{-\mathbf{k}}^{\pm}|\mathbf{E}_0\rangle = -\left\{\begin{array}{c}\cos(dk_{1z}^{+})\\ \sin(dk_{1z}^{-})\end{array}\right\}\frac{4\pi(\mathbf{k}\cdot\mathbf{B}^{\pm})\,ps_{\mathbf{k}}^{\pm}e^{ik_{2z}(z_0 - d)}}{\epsilon_2k_{2z}},$$

$$\langle\tilde{\mathbf{E}}_{\mathbf{k}}^{\pm}|\mathbf{E}_{\mathbf{k}}^{\pm}\rangle = \left\{\frac{(\mathbf{k}\cdot\mathbf{B}^{\pm})^2\left(2dk_{1z}^{\pm}\left(|\mathbf{k}|^2 + (k_{1z}^{\pm})^2\right) + \left(\pm(k_{1z}^{\pm})^2 \mp |\mathbf{k}|^2\right)\sin(2dk_{1z}^{\pm})\right)}{2|\mathbf{k}|^2(k_{1z}^{\pm})^3}\right\}.$$

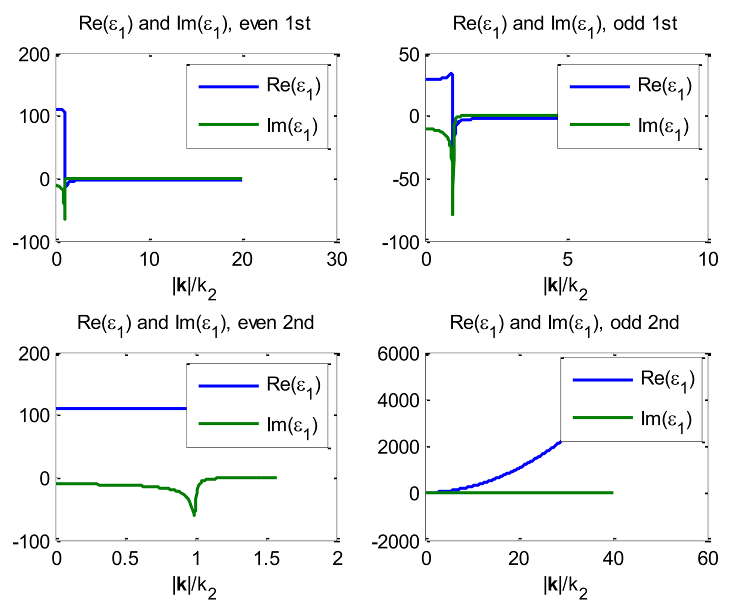
TE modes

$$\mathbf{E}_{\mathbf{k}}^{+} = e^{i\mathbf{k}\cdot\rho}\begin{cases}\mathbf{A}_{xy}^{+}e^{-ik_{2z}z} & \mathbf{r}\in\mathrm{I}\\ \mathbf{B}_{xy}^{+}\cos(k_{1z}^{+}z) & \mathbf{r}\in\mathrm{II}\\ \mathbf{A}_{xy}e^{ik_{2z}z} & \mathbf{r}\in\mathrm{III}\end{cases}$$

$$\mathbf{E}_{\mathbf{k}}^{-} = e^{i\mathbf{k}\cdot\rho}\begin{cases}\mathbf{A}_{xy}^{-}e^{-ik_{2z}z} & \mathbf{r}\in\mathrm{I}\\ \mathbf{B}_{xy}^{-}\sin(k_{1z}^{-}z) & \mathbf{r}\in\mathrm{II}\\ -\mathbf{A}_{xy}^{-}e^{ik_{2z}z} & \mathbf{r}\in\mathrm{III}\end{cases}$$

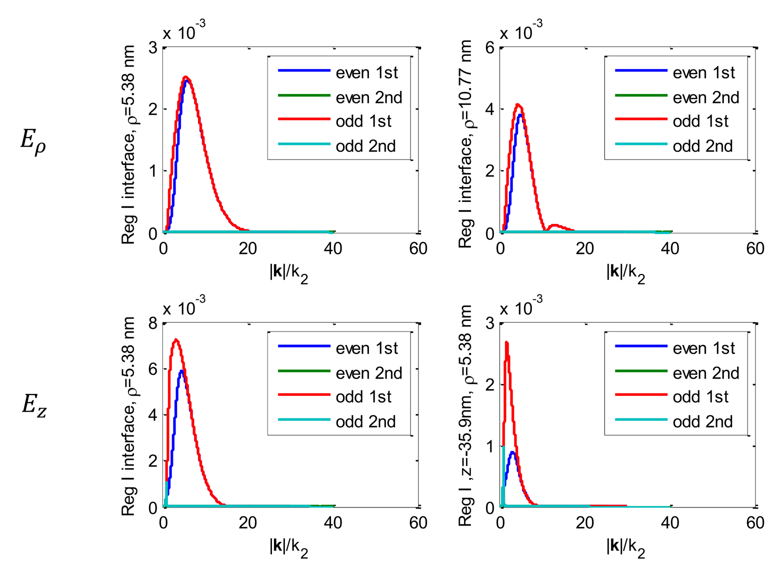
Electric field perpendicular to the plane of incidence.

Eigenvalues as a function of $|\mathbf{k}|$



$$|\mathbf{E}\rangle - |\mathbf{E}_0\rangle = \int\sum_{+,-}\frac{s_{\mathbf{k}}}{s - s_{\mathbf{k}}}|\mathbf{E}_{\mathbf{k}}\rangle\frac{\langle\tilde{\mathbf{E}}_{\mathbf{k}}|\mathbf{E}_0\rangle}{\langle\tilde{\mathbf{E}}_{\mathbf{k}}|\mathbf{E}_{\mathbf{k}}\rangle}d\mathbf{k}.$$

Analytic calculations ➡ $|\mathbf{E}\rangle - |\mathbf{E}_0\rangle = \int\mathbf{F}(|\mathbf{k}|)\,d|\mathbf{k}|$.
The graphs show the integrand as function of $|\mathbf{k}|$ for two different values of z_0 and ρ.



Conclusions

- We calculated the non-quasistatic eigenstates of Maxwell's equations for the microstructure of a slab in a medium.
- Using analytic calculations we expressed the physical electric field as a one dimensional integral which can be calculated fast numerically.
- The higher modes have low contribution to the physical electric field since when the eigenvalues $\epsilon_{1,\mathbf{k}} \to \infty$, $\frac{s_{\mathbf{k}}}{s - s_{\mathbf{k}}} \to 0$.

References

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[3] D. J. Bergman. Perfect imaging of a point charge in the quasistatic regime. Phys. Rev. A **89**, 015801 (2014).
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